ASYMPTOTIC METHODS FOR STOCHASTIC DYNAMICAL SYSTEMS WITH SMALL NON-GAUSSIAN LÉVY NOISE*

HUIJIE QIAO[1] AND JINQIAO DUAN[2]

1. Department of Mathematics, Southeast University
Nanjing, Jiangsu 211189, China
hjqiaogean@yahoo.com.cn
2. Institute for Pure and Applied Mathematics, University of California
Los Angeles, CA 90095
&
Department of Applied Mathematics, Illinois Institute of Technology
Chicago, IL 60616, USA
duan@iit.edu

Abstract. The escape probability is a deterministic tool that quantifies some aspects of stochastic dynamics. The goal of the present work is to analytically examine escape probabilities for dynamical systems driven by symmetric α-stable Lévy motions. Since escape probabilities are solutions of a type of integro-differential equations (i.e., differential equations with nonlocal interactions), asymptotic methods are offered to solve these equations to obtain escape probabilities when noise is sufficiently small. Two examples are presented to illustrate the asymptotic methods, and asymptotic escape probability is compared with numerical simulations.

1. Introduction

Stochastic dynamical systems arise as mathematical models for complex phenomena in biological, geophysical, physical and chemical sciences, under random fluctuations. Unlike the situation for deterministic dynamical systems, an orbit of a stochastic system could vary wildly from one sample to another. It is thus desirable to have efficient tools to quantify stochastic dynamical behaviors. The escape probability is such a tool.

Non-Gaussian random fluctuations are widely observed in various areas such as physics, biology, seismology, electrical engineering and finance [16, 9, 11]. Lévy motions are a class of non-Gaussian processes whose sample paths are discontinuous in time. For a dynamical system driven by Lévy motions, almost all orbits are discontinuous in time. In fact, these orbits are càdlàg (right-continuous with left limit at each time instant), i.e., each of these

Date: January 5, 2012.
AMS Subject Classification(2010): 60H10, 60J75, 35S15, 35B20.
Keywords: Asymptotic methods; stochastic dynamical systems with jumps; escape probability; integro-differential equations; nonlocal interactions.
*This work was partially supported by the NSFC (No. 11001051, 10971225 and 11028102) and the NSF grant DMS-1025422.

orbits has countable jumps in time. Due to these jumps, an orbit could escape an open domain without passing through its boundary. In this case, the *escape probability* is the likelihood that an orbit, starting inside an open domain D, exits this domain first by landing in a target domain U in D^c (the complement of domain D).

For brevity, in this paper we only consider scalar stochastic dynamical systems. Let $\{X_t, t \geqslant 0\}$ be a real-valued Markov process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. Let D be an open domain in $\mathbb{R}$. Define the *exit time*

$$\tau_{D^c} := \inf\{t > 0 : X_t \in D^c\},$$

where D^c is the complement of D in $\mathbb{R}$. Namely, τ_{D^c} is the first time when X_t hits D^c.

When X_t has càdlàg paths which have countable jumps in time, the first hitting of D^c may occur either on the boundary ∂D or somewhere in D^c. For this reason, we take a subset U of the closed set D^c, and define the likelihood that X_t exits firstly from D by landing in the target set U as the escape probability from D to U, denoted by $p(x)$. That is,

$$p(x) = \mathbb{P}\{X_{\tau_{D^c}} \in U\}.$$

If X_t is a solution process of a dynamical system driven by a symmetric α-stable Lévy motion, by [10, 7], the escape probability $p(x)$ solves the following Balayage-Dirichlet "exterior" value problem:

$$\begin{cases} Ap = 0, & x \in D, \\ p|_{D^c} = \varphi, \end{cases} \tag{1}$$

where A is the infinitesimal generator of X_t and φ is defined as follows

$$\varphi(x) = \begin{cases} 1, & x \in U, \\ 0, & x \in D^c \setminus U. \end{cases}$$

However, Eq.(1) is usually an integro-differential equation and it is hard to obtain exact representations for its solutions. Here we use asymptotic methods to examine its solutions. More precisely, **(i)** for a dynamical system driven by a Brownian motion combined with a symmetric α-stable Lévy motion, an asymptotic solution of Eq.(1), or escape probability $p(x)$ from D to U, is obtained by a regular perturbation method; **(ii)** for a dynamical system driven by a symmetric α-stable Lévy motion alone, the escape probability $p(x)$ is obtained by a singular perturbation method.

This paper is arranged as follows. In Section 2, we introduce symmetric α-stable Lévy motions and their infinitesimal generators. In Section 3, a regular perturbation method is applied to examine escape probability for dynamical systems driven jointly by Brownian motion and symmetric α-stable Lévy motions. Escape probabilities for dynamical systems driven by symmetric α-stable Lévy motions alone are studied in Section 4 by a singular perturbation method. Two examples are presented in Section 5. In the Appendix (Section 6), solvability of a type of integro-differential equations is discussed.

2. Preliminaries

In this section, we recall basic concepts and results that will be needed throughout the paper.

Definition 2.1. *A process $L_t = (L_t)_{t\geqslant 0}$ with $L_0 = 0$ a.s. is a Lévy process or Lévy motion if*

(i) L_t has independent increments; that is, $L_t - L_s$ is independent of $L_v - L_u$ if $(u, v) \cap (s, t) = \emptyset$;

(ii) L_t has stationary increments; that is, $L_t - L_s$ has the same distribution as $L_v - L_u$ if $t - s = v - u > 0$;

(iii) L_t is stochastically continuous; and

(iv) L_t is right continuous with left limit.

The characteristic function for L_t is

$$\mathbb{E}\left(\exp\{izL_t\}\right) = \exp\{t\Psi(z)\}, \quad z \in \mathbb{R}.$$

We only consider scalar Lévy motions here. The function $\Psi : \mathbb{R} \to \mathcal{C}$ is called the characteristic exponent of the Lévy process L_t. By Lévy-Khintchine formula, there exist a nonnegative number Q, a measure ν on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}\setminus\{0\}} (|u|^2 \wedge 1)\nu(\mathrm{d}u) < \infty,$$

and also a real number γ such that

$$\Psi(z) \quad = \quad i\gamma z - \frac{1}{2}Qz^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{i\langle z,u\rangle} - 1 - i\langle z, u\rangle 1_{|u|\leqslant 1}\big)\nu(\mathrm{d}u). \tag{2}$$

The measure ν is called a Lévy jump measure, Q is the diffusion, and γ is the drift.

We now introduce a special class of Lévy motions, i.e., the symmetric α-stable Lévy motions L_t^α.

Definition 2.2. *For $\alpha \in (0, 2)$. A scalar symmetric α-stable Lévy motion L_t^α is a Lévy process with characteristic exponent*

$$\Psi(z) = -|z|^\alpha, \quad z \in \mathbb{R}.$$

Thus, for a scalar symmetric α-stable Lévy motion L_t^α, the diffusion $Q = 0$, the drift $\gamma = 0$, and the Lévy jump measure ν is given by

$$\nu(\mathrm{d}u) = \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u,$$

where

$$C_{1,\alpha} = \frac{\alpha\Gamma((1+\alpha)/2)}{2^{1-\alpha}\pi^{1/2}\Gamma(1-\alpha/2)}.$$

Let $\mathcal{C}_0(\mathbb{R})$ be the space of continuous functions f on $\mathbb{R}$ satisfying $\lim\limits_{|x|\to\infty} f(x) = 0$ with norm $\|f\|_{\mathcal{C}_0(\mathbb{R})} = \sup\limits_{x\in\mathbb{R}} |f(x)|$. Let $\mathcal{C}_0^2(\mathbb{R})$ be the set of $f \in \mathcal{C}_0(\mathbb{R})$ such that f is twice

differentiable and the first and second order derivatives of f belong to $\mathcal{C}_0(\mathbb{R})$. Let $\mathcal{L}_\alpha$ be the infinitesimal generator of L_t^α. By [13, Theorem 31.5],

$$(\mathcal{L}_\alpha f)(x) = \int_{\mathbb{R}\setminus\{0\}} \big(f(x+u) - f(x) - f'(x)u1_{|u|\leqslant 1}\big)\nu(\mathrm{d}u),$$

where $f \in \mathcal{C}_0^2(\mathbb{R})$. For any $\varepsilon > 0$, εL_t^α is also a scalar symmetric α-stable Lévy motion, and its Lévy measure $\nu^\varepsilon(B) = \nu(\frac{1}{\varepsilon}B)$ for $B \in \mathscr{B}(\mathbb{R})$ (Borel σ-algebra on $\mathbb{R}$). Thus, its infinitesimal generator is

$$(\mathcal{L}_\alpha^\varepsilon f)(x) := \int_{\mathbb{R}\setminus\{0\}} \big(f(x+u) - f(x) - f'(x)u1_{|u|\leqslant 1}\big)\nu^\varepsilon(\mathrm{d}u), \qquad f \in \mathcal{C}_0^2(\mathbb{R}).$$

Applying the representation of ν, one can obtain that

$$(\mathcal{L}_\alpha^\varepsilon f)(x) = \varepsilon^\alpha \int_{\mathbb{R}\setminus\{0\}} \big(f(x+u) - f(x) - f'(x)u1_{|u|\leqslant 1}\big)\nu(\mathrm{d}u).$$

3. Escape probability of a SDE with Brownian motion and symmetric α-stable Lévy motions

Let $\{W(t)\}_{t\geqslant 0}$ be a scalar standard $\mathcal{F}_t$-adapted Brownian motion, and L_t^α a scalar symmetric α-stable Lévy motion with $\alpha \in (0,2)$ and independent of W_t. Consider the following scalar stochastic differential equation, with drift b, diffusion σ and intensity $\varepsilon(>0)$ of Lévy noise,

$$\left\{\begin{array}{l} \mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t + \varepsilon \mathrm{d}L_t^\alpha, \\ X_0 = x. \end{array}\right. \tag{3}$$

Assume that the drift b and the diffusion $\sigma(\neq 0)$ satisfy the following conditions:

($\mathbf{H}_b$) there exists a constant $C_b > 0$ such that for $x, y \in \mathbb{R}$

$$|b(x) - b(y)| \leqslant C_b|x-y| \cdot \log(|x-y|^{-1} + \mathrm{e});$$

($\mathbf{H}_\sigma$) there exists a constant $C_\sigma > 0$ such that for $x, y \in \mathbb{R}$

$$|\sigma(x) - \sigma(y)|^2 \leqslant C_\sigma|x-y|^2 \cdot \log(|x-y|^{-1} + \mathrm{e}).$$

Under **($\mathbf{H}_b$)** and **($\mathbf{H}_\sigma$)**, it is well known that there exists a unique strong solution to Eq.(3)(see [12]). This solution will be denoted by $X_t(x)$. By Theorem 3.3 in [7], the escape probability $p(x)$ for $X_t(x)$, from $D = (A,B)$ to $U = [B, \infty)$, satisfies the following integro-differential equation

$$b(x)p'(x) + \frac{1}{2}\sigma^2(x)p''(x) + \varepsilon^\alpha \int_{\mathbb{R}\setminus\{0\}} \big(p(x+u) - p(x) - I_{|u|\leqslant 1}p'(x)u\big)\,\nu(\mathrm{d}u) = 0, \quad x \in (A,B), \tag{4}$$

and the 'exterior' conditions

$$p(x)|_{(-\infty,A]} = 0, \qquad p(x)|_{[B,\infty)} = 1. \tag{5}$$

We consider the solution for $p(x)$, when $\varepsilon > 0$ is sufficiently small. Assume that $p(x)$ has the following regular expansion

$$p(x) = p_0(x) + \varepsilon^{\alpha} p_1(x) + \varepsilon^{2\alpha} p_2(x) + \cdots . \tag{6}$$

Substituting (6) into (4) and equating like powers of ε, we obtain a system of equations for the recursive determination of $p_j(x)$. The leading-order equation for $p_0(x)$ is

$$b(x)p_0^{'}(x) + \frac{1}{2}\sigma^2(x)p_0^{''}(x) = 0, \qquad x \in (A, B) \tag{7}$$

with boundary conditions

$$p_0(A) = 0, \qquad p_0(B) = 1. \tag{8}$$

Using the boundary conditions, we solve Problem (7) and (8) to get

$$p_0(x) = \frac{\int_A^x e^{-\int_A^s \phi(u)\mathrm{d}u}\mathrm{d}s}{\int_A^B e^{-\int_A^s \phi(u)\mathrm{d}u}\mathrm{d}s},$$

where $\phi(u) := 2b(u)/\sigma^2(u)$.

Next, the equation for $p_1(x)$ is

$$b(x)p_1^{'}(x) + \frac{1}{2}\sigma^2(x)p_1^{''}(x) + \int_{\mathbb{R}\backslash\{0\}} \Big(p_0(x+u) - p_0(x) - I_{|u|\leqslant 1}p_0^{'}(x)u\Big)\,\nu(\mathrm{d}u) = 0, \qquad x \in (A, B), \tag{9}$$

with boundary conditions

$$p_1(A) = 0, \qquad p_1(B) = 1. \tag{10}$$

Set

$$g(x) := \int_{\mathbb{R}\backslash\{0\}} \Big(p_0(x+u) - p_0(x) - I_{|u|\leqslant 1}p_0^{'}(x)u\Big)\,\nu(\mathrm{d}u).$$

Then Eq.(9) is transformed into the following equation

$$b(x)p_1^{'}(x) + \frac{1}{2}\sigma^2(x)p_1^{''}(x) + g(x) = 0, \qquad x \in (A, B). \tag{11}$$

By solving Problem (11) and (10) we get

$$\begin{aligned} p_1(x) \;\; = \;\; & \int_A^x e^{-\int_A^s \phi(u)\mathrm{d}u} \left(\int_A^s \frac{-2g(u)}{\sigma^2(u)} \cdot e^{\int_A^u \phi(v)\mathrm{d}v}\mathrm{d}u \right) \mathrm{d}s \\ & - p_0(x) \int_A^B e^{-\int_A^s \phi(u)\mathrm{d}u} \left(\int_A^s \frac{-2g(u)}{\sigma^2(u)} \cdot e^{\int_A^u \phi(v)\mathrm{d}v}\mathrm{d}u \right) \mathrm{d}s + p_0(x). \end{aligned}$$

Thus we have an asymptotic expression for escape probability, i.e., solution of Eq.(4), for ε sufficiently small,

$$p(x) \approx p_0(x) + \varepsilon^{\alpha} p_1(x). \tag{12}$$

4. Escape probability of a SDE with symmetric α-stable Lévy motions

Consider the following stochastic differential equation with a symmetric α-stable Lévy motion, with $1 < \alpha < 2$, on $\mathbb{R}$

$$\begin{cases} \mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \varepsilon \mathrm{d}L_t^\alpha, \\ X_0 = x, \end{cases} \tag{13}$$

where the drift b satisfies $(\mathbf{H}_b)$.

By [15, Theorem 3.1], Eq.(13) has a unique solution $X_t(x)$. From Theorem 3.3 in [7], the escape probability $p(x)$, for $X_t(x)$ from $D = (A, B)$ to $U = [B, \infty)$, satisfies the following integro-differential equation

$$b(x)p'(x) + \varepsilon^\alpha \int_{\mathbb{R}\setminus\{0\}} \big(p(x+u) - p(x) - p'(x)u\big)\nu(\mathrm{d}u) = 0, \qquad x \in (A, B), \tag{14}$$

with the 'exterior' conditions

$$p(x)|_{(-\infty, A]} = 0, \tag{15}$$

$$p(x)|_{[B,\infty)} = 1. \tag{16}$$

We now try to construct an asymptotic solution of (14), (15) and (16) for sufficiently small $\varepsilon > 0$. We consider the following four different cases, depending on the dynamical behavior of the corresponding deterministic dynamical system $\dot{x} = b(x)$.

Case 1: $b(x) > 0$ for $x \in (A, B)$. In this case the deterministic dynamical system $\dot{x} = b(x)$ has no equilibrium states and all orbits move to the right.

Thus it is reasonable to require that $p(x) \to 1$ as $\varepsilon \to 0$. We assume that $p(x)$ has the following expansion

$$p(x) = p_0(x) + \varepsilon^\alpha p_1(x) + \varepsilon^{2\alpha} p_2(x) + \cdots. \tag{17}$$

Substituting (17) into (14) and equating like powers of ε, we obtain a system of equations for the recursive determination of $p_j(x)$. The leading-order equation for $p_0(x)$ is

$$b(x)p_0'(x) = 0, \qquad x \in (A, B),$$

and thus $p_0(x) = 1$ for $x \in (A, B)$, because $p(x) \to 1$ as $\varepsilon \to 0$. Since $p_0(x)$ does not satisfy the boundary condition (15), it is necessary to construct a boundary layer correction to $p_0(x)$ near $x = A$.

We introduce a stretched variable

$$\xi = \frac{x - A}{\varepsilon^\beta}$$

with $\beta > 0$ determined later. Defining $F(\xi) = p_0(A + \xi\varepsilon^\beta)$ and inserting it into Eq.(14), we obtain

$$b(A + \xi\varepsilon^\beta)F'(\xi)\varepsilon^{-\beta} + \varepsilon^{\alpha - \alpha\beta} \int_{\mathbb{R}\setminus\{0\}} \big(F(\xi + u) - F(\xi) - F'(\xi)u\big)\nu(\mathrm{d}u) = 0. \tag{18}$$

Set $-\beta = \alpha - \alpha\beta$. That is, we take $\beta = \frac{\alpha}{\alpha-1}$. Multiplying Eq.(18) with ε^{β} and letting $\varepsilon \to 0$, we imply that

$$b(A)F'(\xi) + \int_{\mathbb{R}\backslash\{0\}} \big(F(\xi+u) - F(\xi) - F'(\xi)u\big)\nu(\mathrm{d}u) = 0, \tag{19}$$

with the boundary condition

$$F(\xi) = 0, \quad \xi \leqslant 0, \tag{20}$$

and the matching condition

$$\lim_{\xi\to\infty} F(\xi) = 1. \tag{21}$$

As seen in Appendix at the end of this paper, we know that the system (19)-(21) is solvable, although the solution cannot be expressed in terms of elementary functions. So,

$$p_0(x) = F\left(\frac{x-A}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right).$$

Thus an asymptotic solution of $p(x)$ is, for sufficiently small ε,

$$p(x) \approx F\left(\frac{x-A}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right).$$

Case 2: $b(x) < 0$ for $x \in (A, B)$. Again, in this case the deterministic dynamical system $\dot{x} = b(x)$ has no equilibrium states and all orbits move to the left.

Thus as $\varepsilon \to 0$, $p(x) \to 0$. We assume that $p(x)$ has the following expansion

$$p(x) = p_0(x) + \varepsilon^{\alpha} p_1(x) + \varepsilon^{2\alpha} p_2(x) + \cdots. \tag{22}$$

Similar to **Case 1**, we obtain the leading-order equation for $p_0(x)$

$$b(x)p_0^{'}(x) = 0, \qquad x \in (A, B).$$

So, $p_0(x) = 0$ for $x \in (A, B)$, because $p(x) \to 0$ as $\varepsilon \to 0$. Since $p_0(x)$ does not satisfy the boundary condition (16), it is necessary to construct a boundary layer correction to $p_0(x)$ near $x = B$.

We introduce a stretched variable

$$\varsigma = \frac{B-x}{\varepsilon^{\beta}},$$

where β is the same as one in **Case 1**. Defining $G(\varsigma) = p_0(B - \varsigma\varepsilon^{\beta})$ and inserting it into Eq.(14), we obtain

$$-b(B-\varsigma\varepsilon^{\beta})G'(\varsigma)\varepsilon^{-\beta} + \varepsilon^{\alpha-\alpha\beta}\int_{\mathbb{R}\backslash\{0\}} \big[G(\varsigma-u) - G(\varsigma) - G'(\varsigma)(-u)\big]\nu(\mathrm{d}u) = 0. \tag{23}$$

Multiplying Eq.(23) with ε^{β} and letting $\varepsilon \to 0$, we obtain

$$-b(B)G'(\varsigma) + \int_{\mathbb{R}\backslash\{0\}} \big(G(\varsigma+u) - G(\varsigma) - G'(\varsigma)u\big)\nu(\mathrm{d}u) = 0, \tag{24}$$

with the boundary condition

$$G(\varsigma) = 1, \quad \varsigma \leqslant 0, \tag{25}$$

and the matching condition

$$\lim_{\varsigma \to \infty} G(\varsigma) = 0. \tag{26}$$

As seen in Appendix, the system (24)-(26) is solvable, although the solution cannot be expressed in terms of elementary functions. So,

$$p_0(x) = G\left(\frac{B-x}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right).$$

Thus we obtain an asymptotic solution of Eq.(14)

$$p(x) \approx G\left(\frac{B-x}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right).$$

Case 3: There exists a $\bar{x} \in (A, B)$ such that $b(\bar{x}) = 0$ and $b'(\bar{x}) > 0$. In this case the deterministic dynamical system $\dot{x} = b(x)$ has one *unstable* equilibrium state $\bar{x}$. Then as $\varepsilon \to 0$, $p(x) \to 1$ for $\bar{x} < x \leqslant B$ and $p(x) \to 0$ for $A \leqslant x < \bar{x}$. We assume that $p(x)$ has the following expansion

$$p(x) = p_0(x) + \varepsilon^{\alpha} p_1(x) + \varepsilon^{2\alpha} p_2(x) + \cdots. \tag{27}$$

As in **Case 1**, we obtain that the leading-order equation for $p_0(x)$ is

$$b(x)p_0^{'}(x) = 0, \qquad x \in (A, B).$$

So,

$$p_0(x) = \begin{cases} 1, & \bar{x} < x \leqslant B, \\ 0, & A \leqslant x < \bar{x}. \end{cases}$$

Although $p_0(x)$ partially satisfies the 'exterior' conditions (15) and (16), the value of $p_0(x)$ around $\bar{x}$ is unknown. Therefore, it is necessary to construct an internal boundary layer correction to $p_0(x)$ near $x = \bar{x}$.

We introduce a stretched variable

$$\eta = \frac{x - \bar{x}}{\varepsilon}.$$

Define $H(\eta) = p_0(\bar{x} + \eta\varepsilon)$ and insert it into Eq.(14). Then Eq.(14) becomes

$$b(\bar{x} + \eta\varepsilon)H'(\eta)\varepsilon^{-1} + \int_{\mathbb{R}\setminus\{0\}} \big(H(\eta+u) - H(\eta) - H'(\eta)u\big)\nu(\mathrm{d}u) = 0.$$

Letting $\varepsilon \to 0$ and using the L'Hospital's rule, we get

$$b'(\bar{x})\eta H'(\eta) + \int_{\mathbb{R}\setminus\{0\}} \big(H(\eta+u) - H(\eta) - H'(\eta)u\big)\nu(\mathrm{d}u) = 0, \tag{28}$$

with the matching conditions

$$\lim_{\eta \to -\infty} H(\eta) = 0, \tag{29}$$

$$\lim_{\eta \to \infty} H(\eta) = 1. \tag{30}$$

As seen in Appendix, Eq.(28) is solvable. So,

$$p_0(x) = H\left(\frac{x - \bar{x}}{\varepsilon}\right).$$

Thus we obtain an asymptotic solution of Eq.(14)

$$p(x) \approx H\left(\frac{x - \bar{x}}{\varepsilon}\right).$$

Case 4: There exists a $\bar{x} \in (A, B)$ such that $b(\bar{x}) = 0$ and $b'(\bar{x}) < 0$. In this case the deterministic dynamical system $\dot{x} = b(x)$ has one *stable* equilibrium state $\bar{x}$. We further require that $b(A)b(B) \neq 0$, i.e., A and B are not equilibrium states for $\dot{x} = b(x)$.

We assume that $p(x)$ has the following expansion

$$p(x) = p_0(x) + \varepsilon^{\alpha} p_1(x) + \varepsilon^{2\alpha} p_2(x) + \cdots . \tag{31}$$

As in **Case 1**, we obtain the leading-order equation for $p_0(x)$

$$b(x) p_0^{'}(x) = 0, \qquad x \in (A, B).$$

So, $p_0(x) = C$ for $x \in (A, B)$. Because of not knowing at which endpoint there will be a boundary layer correction, we construct asymptotic approximations near both endpoints. If there is a boundary layer correction near $x = A$ and $x = B$, respectively, as in **Case 1** and **Case 2**, we get near $x = A$

$$p_0(x) = C \cdot F\left(\frac{x - A}{\varepsilon^{\frac{\alpha}{\alpha - 1}}}\right),$$

and near $x = B$

$$p_0(x) = C + (1 - C) \cdot G\left(\frac{B - x}{\varepsilon^{\frac{\alpha}{\alpha - 1}}}\right).$$

Thus we have an asymptotic solution

$$p(x) \approx C \cdot F\left(\frac{x - A}{\varepsilon^{\frac{\alpha}{\alpha - 1}}}\right) + (1 - C) \cdot G\left(\frac{B - x}{\varepsilon^{\frac{\alpha}{\alpha - 1}}}\right).$$

Since F, G cannot be expressed in terms of elementary functions, it is hard to determine C. But for a concrete example in the next section, Example 5.2, we introduce a method to determine the value of C.

5. Examples

In this section we consider two examples. Example 5.1 and Example 5.2 correspond to our methods in Section 3 and Section 4, respectively.

Example 5.1. *Consider the following scalar* SDE *with a Brownian motion and a symmetric α-stable Lévy motion:*

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}W_t + \varepsilon \mathrm{d}L_t^\alpha, \\ X_0 = x. \end{cases}$$

The unique solution is denoted as $X_t(x)$. We take $(A, B) = (-1, 1)$ and $[B, \infty) = [1, \infty)$. The escape probability $p(x)$, for $X_t(x)$ from $(-1, 1)$ to $[1, \infty)$, satisfies the following integro-differential equation

$$\frac{1}{2}p''(x) + \varepsilon^\alpha \int_{\mathbb{R}\setminus\{0\}} \big(p(x+u) - p(x) - I_{|u|\leqslant 1} p'(x)u\big)\, \nu(\mathrm{d}u) = 0, \qquad x \in (-1, 1), \tag{32}$$

and the exterior conditions

$$p(x)|_{(-\infty,-1]} = 0, \qquad p(x)|_{[1,\infty)} = 1.$$

We seek an asymptotic solution of $p(x)$ as follows

$$p(x) \approx p_0(x) + \varepsilon^\alpha p_1(x),$$

where

$$p_0(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1, \\ 0, & x \leqslant -1, \\ 1, & x \geqslant 1, \end{cases}$$

and

$$p_1(x) = \begin{cases} \frac{C_{1,\alpha}}{(-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)}\big[(1-x)^{3-\alpha} - 2^{3-\alpha} + (3-\alpha)2^{2-\alpha}(x+1) - (1+x)^{3-\alpha}\big] & \\ -\frac{x+1}{2}\frac{C_{1,\alpha}}{(-\alpha)(2-\alpha)(3-\alpha)}2^{3-\alpha} + \frac{x+1}{2}, & -1 < x < 1, \\ 0, & x \leqslant -1, \\ 1, & x \geqslant 1. \end{cases}$$

Next we use a numerical method to study Eq.(32). By the exterior conditions, Eq.(32) is changed into

$$\begin{aligned} &\frac{1}{2}p''(x) - \varepsilon^\alpha C_{1,\alpha}\frac{1-(1+x)^{-\alpha+1}}{\alpha-1}p'(x) - \varepsilon^\alpha C_{1,\alpha}\left(\frac{(1+x)^{-\alpha}}{\alpha} + \frac{(1-x)^{-\alpha}}{\alpha}\right)p(x) \\ &+\varepsilon^\alpha C_{1,\alpha}\int_{-x-1}^{1}\frac{p(x+u)-p(x)-p'(x)u}{|u|^{1+\alpha}}\mathrm{d}u + \varepsilon^\alpha C_{1,\alpha}\int_{1}^{-x+1}\frac{p(x+u)-p(x)}{|u|^{1+\alpha}}\mathrm{d}u \\ &+\varepsilon^\alpha C_{1,\alpha}\frac{(1-x)^{-\alpha}}{\alpha} = 0, \end{aligned}$$

for $-1 < x < 0$ and

$$\begin{aligned} &\frac{1}{2}p''(x) - \varepsilon^\alpha C_{1,\alpha}\frac{1-(1-x)^{-\alpha+1}}{1-\alpha}p'(x) - \varepsilon^\alpha C_{1,\alpha}\left(\frac{(1+x)^{-\alpha}}{\alpha} + \frac{(1-x)^{-\alpha}}{\alpha}\right)p(x) \\ &+\varepsilon^\alpha C_{1,\alpha}\int_{-x-1}^{-1}\frac{p(x+u)-p(x)}{|u|^{1+\alpha}}\mathrm{d}u + \varepsilon^\alpha C_{1,\alpha}\int_{-1}^{-x+1}\frac{p(x+u)-p(x)-p'(x)u}{|u|^{1+\alpha}}\mathrm{d}u \end{aligned}$$

$$+\varepsilon^\alpha C_{1,\alpha}\frac{(1-x)^{-\alpha}}{\alpha}=0,$$

for $0\leqslant x<1$. *For any* $K\in\mathbb{N}$, *divid equally the interval* $[-2,2]$ *into* $4K$ *subintervals and let* $x_k:=kh$ *and* $P_k:=p(x_k)$ *for* $-2K\leqslant k\leqslant 2K$, *where* $h=\frac{1}{K}$. *Thus, by central difference for derivatives and punched-hole trapezoidal rule, one can obtain*

$$\begin{aligned}
&\frac{1}{2}\frac{P_{k-1}-2P_k+P_{k+1}}{h^2}-\varepsilon^\alpha C_{1,\alpha}\frac{1-(1+x_k)^{-\alpha+1}}{\alpha-1}\frac{P_{k+1}-P_{k-1}}{2h}\\
&-\varepsilon^\alpha C_{1,\alpha}\left(\frac{(1+x_k)^{-\alpha}}{\alpha}+\frac{(1-x_k)^{-\alpha}}{\alpha}\right)P_k+\varepsilon^\alpha C_{1,\alpha}\frac{(1-x_k)^{-\alpha}}{\alpha}\\
&+\varepsilon^\alpha C_{1,\alpha}h{\sum}'^{K}_{i=-K-k}\frac{P_{i+k}-P_k-(P_{k+1}-P_{k-1})x_i/2h}{|x_i|^{1+\alpha}}\\
&+\varepsilon^\alpha C_{1,\alpha}h{\sum}'^{K-k}_{i=K}\frac{P_{i+k}-P_k}{|x_i|^{1+\alpha}}=0,
\end{aligned}$$

for $k=-K+1,\cdots,-1$ *and*

$$\begin{aligned}
&\frac{1}{2}\frac{P_{k-1}-2P_k+P_{k+1}}{h^2}-\varepsilon^\alpha C_{1,\alpha}\frac{1-(1-x_k)^{-\alpha+1}}{1-\alpha}\frac{P_{k+1}-P_{k-1}}{2h}\\
&-\varepsilon^\alpha C_{1,\alpha}\left(\frac{(1+x_k)^{-\alpha}}{\alpha}+\frac{(1-x_k)^{-\alpha}}{\alpha}\right)P_k+\varepsilon^\alpha C_{1,\alpha}\frac{(1-x_k)^{-\alpha}}{\alpha}\\
&+\varepsilon^\alpha C_{1,\alpha}h{\sum}'^{K-k}_{i=-K}\frac{P_{i+k}-P_k-(P_{k+1}-P_{k-1})x_i/2h}{|x_i|^{1+\alpha}}\\
&+\varepsilon^\alpha C_{1,\alpha}h{\sum}'^{-K}_{i=-K-k}\frac{P_{i+k}-P_k}{|x_i|^{1+\alpha}}=0,
\end{aligned}$$

for $k=0,1,\cdots,K-1$. *Here the modified summation symbol "*$\sum'$*" means that the quantities corresponding to the two end summation indices are multiplied by* $1/2$ *when they are added. By the error analysis in* [3], *the following scheme have* 2*nd-order accuracy for any* $0<\alpha<2$

$$\begin{aligned}
&C_h\frac{P_{k-1}-2P_k+P_{k+1}}{h^2}-\varepsilon^\alpha C_{1,\alpha}\frac{1-(1+x_k)^{-\alpha+1}}{\alpha-1}\frac{P_{k+1}-P_{k-1}}{2h}\\
&-\varepsilon^\alpha C_{1,\alpha}\left(\frac{(1+x_k)^{-\alpha}}{\alpha}+\frac{(1-x_k)^{-\alpha}}{\alpha}\right)P_k+\varepsilon^\alpha C_{1,\alpha}\frac{(1-x_k)^{-\alpha}}{\alpha}\\
&+\varepsilon^\alpha C_{1,\alpha}h{\sum}'^{K}_{i=-K-k}\frac{P_{i+k}-P_k-(P_{k+1}-P_{k-1})x_i/2h}{|x_i|^{1+\alpha}}\\
&+\varepsilon^\alpha C_{1,\alpha}h{\sum}'^{K-k}_{i=K}\frac{P_{i+k}-P_k}{|x_i|^{1+\alpha}}=0,
\end{aligned}\tag{33}$$

for $k=-K+1,\cdots,-1$ *and*

$$C_h\frac{P_{k-1}-2P_k+P_{k+1}}{h^2}-\varepsilon^\alpha C_{1,\alpha}\frac{1-(1-x_k)^{-\alpha+1}}{1-\alpha}\frac{P_{k+1}-P_{k-1}}{2h}$$

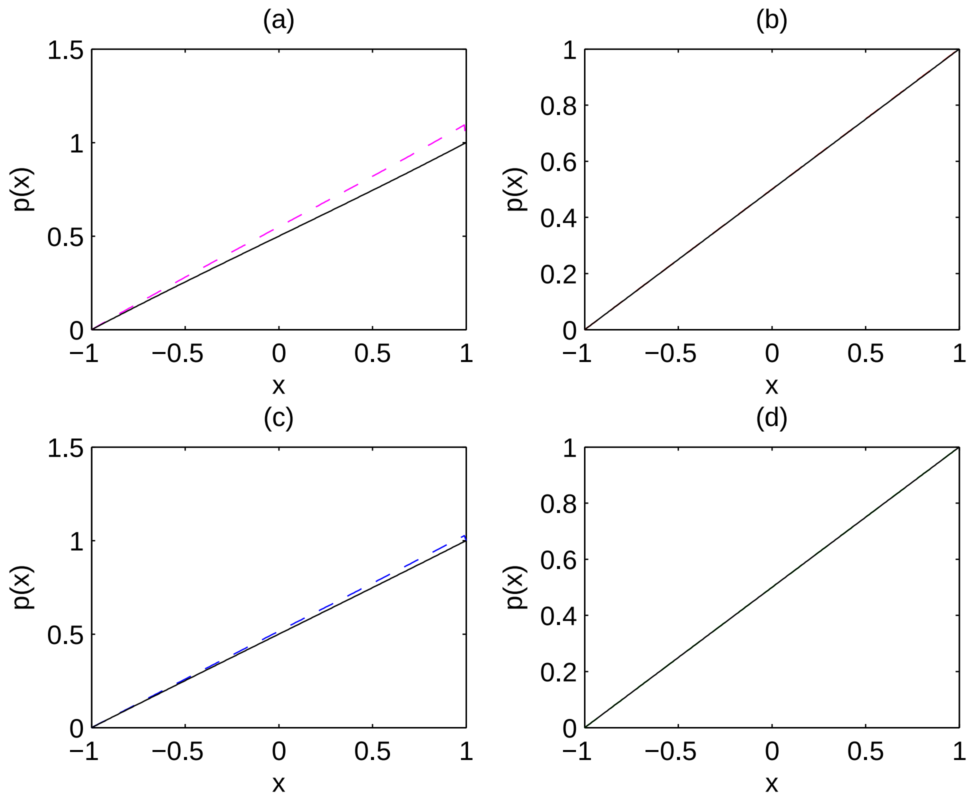

FIGURE 1. Comparison between the asymptotic solution and the numerical solution of Eq.(32) for small ε. (a) $\alpha = 0.5, \varepsilon = 0.01$. The asymptotic solution is shown with dashed line while the numerical solution is displayed with solid line. (b) $\alpha = 1.5, \varepsilon = 0.01$. (c) $\alpha = 0.5, \varepsilon = 0.001$. (d) $\alpha = 1.5, \varepsilon = 0.001$.

$$
\begin{aligned}
&-\varepsilon^{\alpha} C_{1,\alpha} \left(\frac{(1+x_k)^{-\alpha}}{\alpha} + \frac{(1-x_k)^{-\alpha}}{\alpha} \right) P_k + \varepsilon^{\alpha} C_{1,\alpha} \frac{(1-x_k)^{-\alpha}}{\alpha} \\
&+\varepsilon^{\alpha} C_{1,\alpha} h {\sum}'^{K-k}_{i=-K} \frac{P_{i+k} - P_k - (P_{k+1} - P_{k-1}) x_i / 2h}{|x_i|^{1+\alpha}} \\
&+\varepsilon^{\alpha} C_{1,\alpha} h {\sum}'^{-K}_{i=-K-k} \frac{P_{i+k} - P_k}{|x_i|^{1+\alpha}} = 0,
\end{aligned} \tag{34}
$$

for $k = 0, 1, \cdots, K-1$, where $C_h = \frac{1}{2} - \varepsilon^{\alpha} C_{1,\alpha} \zeta(\alpha - 1) h^{2-\alpha}$ and ζ is the Riemann zeta function. By solving these linear equations (33) and (34), we obtain the numerical solution of Eq.(32).

Figure 1 shows that if ε is fixed and α turns large, the difference between the asymptotic solution and the numerical solution of Eq.(32) will become small; if α is fixed and ε

becomes large, the difference will turn large, because the asymptotic solution is for sufficiently small ε.

Example 5.2. *Consider the following scalar* SDE *with a symmetric* α*-stable Lévy motion, with* $1 < \alpha < 2$,

$$\begin{cases} \mathrm{d}X_t = -X_t\,\mathrm{d}t + \varepsilon \mathrm{d}L_t^{\alpha}, \\ X_0 = x. \end{cases}$$

The unique solution is denoted as $X_t(x)$. *We take* $(A, B) = (-1, 1)$ *and* $[B, \infty) = [1, \infty)$. *The escape probability* $p(x)$, *for* $X_t(x)$ *from* $(-1, 1)$ *to* $[1, \infty)$, *satisfies the following integro-differential equation*

$$-xp'(x) + \varepsilon^{\alpha} \int_{\mathbb{R}\setminus\{0\}} (p(x+u) - p(x) - p'(x)u)\,\nu(\mathrm{d}u) = 0, \qquad x \in (-1, 1), \tag{35}$$

with the exterior conditions

$$p(x)|_{(-\infty,-1]} = 0, \qquad p(x)|_{[1,\infty)} = 1. \tag{36}$$

Since $b(x) = -x, b(0) = 0, b'(0) < 0, b(-1)b(1) = -1 \neq 0$, *by the result of* **Case 4** *in Section 4, an asymptotic solution of* $p(x)$ *is given by*

$$p(x) \approx C \cdot F\left(\frac{x+1}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right) + (1 - C) \cdot G\left(\frac{1-x}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right). \tag{37}$$

Specially, take the Lévy measure

$$\nu(\mathrm{d}u) = \frac{\kappa}{|u|^{1+\alpha}} \cdot 1_{|u|\leqslant 1}\mathrm{d}u,$$

where $\kappa > 0$ *is a constant([16]). Thus, the function* F *can be given explicitly by*

$$F(x) = \begin{cases} 1 - e^{-\gamma x}, & x > 0, \\ 0, & x \leqslant 0, \end{cases}$$

where $\gamma > 0$ *satisfies the following integral equation*

$$\gamma - \int_{-1}^{1} \left(e^{-\gamma u} - 1 - (-\gamma)u\right) \frac{\kappa}{|u|^{1+\alpha}}\mathrm{d}u = 0.$$

By the relation between F *and* G, *we can obtain*

$$G(x) = \begin{cases} e^{-\gamma x}, & x > 0, \\ 1, & x \leqslant 0. \end{cases}$$

So, the asymptotic solution of Eq.(35) is given by

$$p(x) \approx C\left(1 - \exp\left\{-\gamma\left(\frac{x+2}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right)\right\}\right) + (1 - C)\exp\left\{-\gamma\left(\frac{1-x}{\varepsilon^{\frac{\alpha}{\alpha-1}}}\right)\right\}.$$

To determine C, we multiply Eq.(35) by the solution $\rho(x)$ of the steady Fokker-Planck equation

$$-(-x\rho(x))' + \varepsilon^{\alpha} \int_{\mathbb{R}\backslash\{0\}} (\rho(x+u) - \rho(x) - \rho'(x)u)\, \nu(\mathrm{d}u) = 0 \tag{38}$$

and integrate over $(-1,1)$, to obtain

$$\int_{-1}^{1} \big(-x\rho(x)\big) p'(x)\mathrm{d}x + \varepsilon^{\alpha} \int_{-1}^{1} \rho(x)\mathrm{d}x \int_{\mathbb{R}\backslash\{0\}} (p(x+u) - p(x) - p'(x)u)\, \nu(\mathrm{d}u) = 0. \tag{39}$$

To (39), by integration by parts and using (38), we get

$$\begin{aligned} &-\rho(1) - \varepsilon^{\alpha} \int_{-1}^{1} p(x)\mathrm{d}x \int_{\mathbb{R}\backslash\{0\}} (\rho(x+u) - \rho(x) - \rho'(x)u)\, \nu(\mathrm{d}u) \\ &+\varepsilon^{\alpha} \int_{-1}^{1} \rho(x)\mathrm{d}x \int_{\mathbb{R}\backslash\{0\}} (p(x+u) - p(x) - p'(x)u)\, \nu(\mathrm{d}u) = 0. \end{aligned}$$

Applying Cauchy principal value and (36), we have

$$\begin{aligned} & -\frac{\varepsilon^{-\alpha}\rho(1)}{C_{1,\alpha}} + \int_{-1}^{1} \rho(x) \frac{(1-x)^{-\alpha}}{\alpha} \mathrm{d}x \\ = \;& \int_{-1}^{1} p(x)\mathrm{d}x \left[\int_{-\infty}^{-1} \frac{\rho(y)}{|y-x|^{1+\alpha}} \mathrm{d}y + \int_{1}^{\infty} \frac{\rho(y)}{|y-x|^{1+\alpha}} \mathrm{d}y \right]. \end{aligned} \tag{40}$$

By [1, Proposition 3.2], *the Fourier transform of $\rho(k)$ is given by*

$$\hat{\rho}(k) = \exp\{-\frac{\varepsilon^{\alpha}}{\alpha}|k|^{\alpha}\}.$$

Replacing $\rho(x)$ and $p(x)$ by $\frac{1}{2\pi}\int_{\mathbb{R}} e^{ixk}\hat{\rho}(k)\mathrm{d}k$ and (37), respectively, and letting $\varepsilon \to 0$, we can obtain C from (40).

6. Appendix

In this appendix, we consider the system (19)-(20)-(21), arising in the analysis in this paper, including a useful identity, solvability condition, and the matching condition for the asymptotic solution.

6.1. **A useful integral identity.**

First, we prove a useful integral identity:

$$\int_{0}^{\infty} \left(e^{-\lambda u} - 1 - (-\lambda)u\right) \frac{1}{u^{1+\alpha}} \mathrm{d}u = \frac{\lambda^{\alpha}\Gamma(2-\alpha)}{\alpha(\alpha-1)}. \tag{41}$$

Proof. By integration by parts,

$$\begin{aligned} \int_{0}^{\infty} \left(e^{-\lambda u} - 1 - (-\lambda)u\right) \frac{1}{u^{1+\alpha}} \mathrm{d}u &= -\frac{1}{\alpha} \int_{0}^{\infty} \left(e^{-\lambda u} - 1 - (-\lambda)u\right) \mathrm{d}u^{-\alpha} \\ &= -\frac{\lambda}{\alpha} \int_{0}^{\infty} u^{-\alpha}(e^{-\lambda u} - 1)\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda^2}{\alpha(\alpha-1)}\int_0^\infty e^{-\lambda u}u^{-\alpha+1}\mathrm{d}u \\
&= \frac{\lambda^\alpha}{\alpha(\alpha-1)}\int_0^\infty e^{-u}u^{-\alpha+1}\mathrm{d}u \\
&= \frac{\lambda^\alpha}{\alpha(\alpha-1)}\Gamma(2-\alpha).
\end{aligned}
$$

This completes the proof. □

6.2. **Solvability.** Next, we consider solvability of the system (19)-(20)-(21). By [4, 5, 6], we know that Eq.(19) is solvable, but the solution cannot be expressed in terms of elementary functions.

6.3. **Asymptotic behavior.** Finally, we examine the asymptotic behavior of the solution to the system (19)-(20)-(21).

For $\xi > 0$, using (20), Eq.(19) is transformed to

$$
\begin{aligned}
& b(A)F'(\xi) + \int_{-\infty}^{-\xi}\left(-F(\xi) - F'(\xi)u\right)\nu(\mathrm{d}u) \\
+ & \int_{-\xi}^{\infty}\left(F(\xi+u) - F(\xi) - F'(\xi)u\right)\nu(\mathrm{d}u) = 0. \qquad (42)
\end{aligned}
$$

Denote the Laplace transformation of $F(\xi)$ by $\tilde{F}(\lambda)$, i.e.

$$
\tilde{F}(\lambda) := \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi, \quad \lambda > 0.
$$

Now applying the Laplace transformation to Eq.(42), we get that for the first term of its left hand side (LHS)

$$
\begin{aligned}
\int_0^\infty e^{-\lambda\xi}b(A)F'(\xi)\mathrm{d}\xi &= b(A)\lambda\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \\
&= b(A)\lambda\tilde{F}(\lambda). \qquad (43)
\end{aligned}
$$

For the second term in the LHS of Eq.(42),

$$
\begin{aligned}
& \int_0^\infty e^{-\lambda\xi}\left(\int_{-\infty}^{-\xi}\left(-F(\xi) - F'(\xi)u\right)\nu(\mathrm{d}u)\right)\mathrm{d}\xi \\
= & -\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_{-\infty}^{-\xi}\nu(\mathrm{d}u) - \int_0^\infty e^{-\lambda\xi}F'(\xi)\mathrm{d}\xi\int_{-\infty}^{-\xi}u\nu(\mathrm{d}u) \\
= & -\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_{-\infty}^{-\xi}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u - \int_0^\infty e^{-\lambda\xi}F'(\xi)\mathrm{d}\xi\int_{-\infty}^{-\xi}u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
= & -\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_{-\infty}^{-\xi}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u - \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_{-\infty}^{-\xi}\lambda u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u
\end{aligned}
$$

$$+\int_0^\infty e^{-\lambda\xi}F(\xi)\frac{C_{1,\alpha}}{\xi^\alpha}\mathrm{d}\xi. \tag{44}$$

Now we deal with the third term in the LHS of Eq.(42). Firstly, let $\xi+u=\bar{u}$ and then

$$\begin{aligned}&\int_{-\xi}^\infty \left(F(\xi+u)-F(\xi)-F'(\xi)u\right)\nu(\mathrm{d}u)\\ =\ &\int_0^\infty \left(F(\bar{u})-F(\xi)-F'(\xi)(\bar{u}-\xi)\right)\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u}.\end{aligned} \tag{45}$$

Secondly, applying the Laplace transformation to Eq.(45), we have

$$\begin{aligned}&\int_0^\infty e^{-\lambda\xi}\left(\int_0^\infty \left(F(\bar{u})-F(\xi)-F'(\xi)(\bar{u}-\xi)\right)\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u}\right)\mathrm{d}\xi\\ =:\ &I_1-I_2-I_3,\end{aligned} \tag{46}$$

where

$$I_1=\int_0^\infty e^{-\lambda\xi}\mathrm{d}\xi\int_0^\infty F(\bar{u})\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u},$$
$$I_2=\int_0^\infty e^{-\lambda\xi}\mathrm{d}\xi\int_0^\infty F(\xi)\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u},$$
$$I_3=\int_0^\infty e^{-\lambda\xi}\mathrm{d}\xi\int_0^\infty F'(\xi)(\bar{u}-\xi)\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u}.$$

For I_1, interchanging the order of integrals, we obtain

$$\begin{aligned}I_1\ &=\ \int_0^\infty F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty e^{-\lambda\xi}\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\xi\\ &=\ \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty e^{-\lambda(\xi-\bar{u})}\frac{C_{1,\alpha}}{|\xi-\bar{u}|^{1+\alpha}}\mathrm{d}\xi\\ &=\ \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^\infty e^{-\lambda u}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u\\ &=\ \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^0 e^{-\lambda u}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u\\ &\quad+\int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty e^{-\lambda u}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u.\end{aligned} \tag{47}$$

We now treat I_2 as follows:

$$\begin{aligned}I_2\ &=\ \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_0^\infty \frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u}\\ &=\ \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty \frac{C_{1,\alpha}}{|\xi-\bar{u}|^{1+\alpha}}\mathrm{d}\xi\\ &=\ \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^\infty \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^0 \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad +\int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u. \qquad (48)
\end{aligned}
$$

For I_3, we have

$$
\begin{aligned}
I_3 &= \int_0^\infty e^{-\lambda\xi}F'(\xi)\mathrm{d}\xi\int_0^\infty (\bar{u}-\xi)\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u} \\
&= \int_0^\infty e^{-\lambda\xi}F'(\xi)\mathrm{d}\xi\int_{-\xi}^\infty u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&= \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_{-\xi}^\infty \lambda u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad +\int_0^\infty e^{-\lambda\xi}F(\xi)\frac{C_{1,\alpha}}{\xi^\alpha}\mathrm{d}\xi, \qquad (49)
\end{aligned}
$$

where the Cauchy principal value is used in the last equality.

Substituting (47), (48) and (49) into (46), and using (41), we obtain

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda\xi}\left(\int_0^\infty \left(F(\bar{u})-F(\xi)-F'(\xi)(\bar{u}-\xi)\right)\frac{C_{1,\alpha}}{|\bar{u}-\xi|^{1+\alpha}}\mathrm{d}\bar{u}\right)\mathrm{d}\xi \\
&= \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^0 e^{-\lambda u}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u+\int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty e^{-\lambda u}\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad -\int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^0 \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u-\int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad -\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_{-\xi}^\infty \lambda u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u-\int_0^\infty e^{-\lambda\xi}F(\xi)\frac{C_{1,\alpha}}{\xi^\alpha}\mathrm{d}\xi \\
&= \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^0 \left(e^{-\lambda u}-1-(-\lambda)u\right)\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad +\int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_0^\infty \left(e^{-\lambda u}-1-(-\lambda)u\right)\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad -2\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_\xi^\infty \lambda u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u-\int_0^\infty e^{-\lambda\xi}F(\xi)\frac{C_{1,\alpha}}{\xi^\alpha}\mathrm{d}\xi \\
&= \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u}\int_{-\bar{u}}^0 \left(e^{-\lambda u}-1-(-\lambda)u\right)\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
&\quad -2\int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi\int_\xi^\infty \lambda u\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u-\int_0^\infty e^{-\lambda\xi}F(\xi)\frac{C_{1,\alpha}}{\xi^\alpha}\mathrm{d}\xi \\
&\quad +\frac{\lambda^\alpha\Gamma(2-\alpha)}{\alpha(\alpha-1)}C_{1,\alpha}\tilde{F}(\lambda). \qquad (50)
\end{aligned}
$$

Combining (43), (44) and (50), we get

$$
\begin{aligned}
& b(A)\lambda\tilde{F}(\lambda) - \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \int_{-\infty}^{-\xi} \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u - \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \int_\xi^\infty \lambda u \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
& + \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u} \int_{-\bar{u}}^0 \left(e^{-\lambda u} - 1 - (-\lambda)u\right) \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u + \frac{\lambda^\alpha\Gamma(2-\alpha)}{\alpha(\alpha-1)}C_{1,\alpha}\tilde{F}(\lambda) \\
= & \left(b(A)\lambda + \frac{\lambda^\alpha\Gamma(2-\alpha)}{\alpha(\alpha-1)}C_{1,\alpha}\right)\tilde{F}(\lambda) - \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \int_\xi^\infty (1+\lambda u)\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
& + \int_0^\infty e^{-\lambda\bar{u}}F(\bar{u})\mathrm{d}\bar{u} \int_{-\bar{u}}^0 \left(e^{-\lambda u} - 1 - (-\lambda)u\right) \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
= & \left(b(A)\lambda + \frac{\lambda^\alpha\Gamma(2-\alpha)}{\alpha(\alpha-1)}C_{1,\alpha}\right)\tilde{F}(\lambda) - \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \int_\xi^\infty (1+\lambda u)\frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
& + \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \int_0^\xi \left(e^{\lambda u} - 1 - \lambda u\right) \frac{C_{1,\alpha}}{|u|^{1+\alpha}}\mathrm{d}u \\
= & \ 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{F}(\lambda) = & \frac{\alpha(\alpha-1)C_{1,\alpha}}{\alpha(\alpha-1)b(A)\lambda + \lambda^\alpha\Gamma(2-\alpha)C_{1,\alpha}} \cdot \int_0^\infty e^{-\lambda\xi}F(\xi)\mathrm{d}\xi \Bigg[\int_\xi^\infty \frac{1+\lambda u}{|u|^{1+\alpha}}\mathrm{d}u \\
& - \int_0^\xi \frac{e^{\lambda u} - 1 - \lambda u}{|u|^{1+\alpha}}\mathrm{d}u\Bigg].
\end{aligned}
$$

Thus as $\lambda \to 0$, $\tilde{F}(\lambda) \to \infty$, which implies that $F(\xi)$ approaches a constant for $\xi \gg 1$ and satisfies the matching condition (21).

Acknowledgement. We thank Xingye Kan for help with Example 5.1. This work was done while Huijie Qiao was visiting the Institute for Pure and Applied Mathematics (IPAM), Los Angeles.

References

[1] S. Albeverio, B. Rüdiger and J. Wu: Invariant measures and symmetry property of Lévy type operators, *Potential Analysis,* 13(2000)147-168.

[2] D. Applebaum: *Lévy Processes and Stochastic Calculus*, Second Edition, Cambridge Univ. Press, Cambridge, 2009.

[3] T. Gao, J. Duan, X. Li and R. Song, Mean exit time and escape probability for dynamical systems driven by Lévy noise. Submitted to *Methodology and Computing in Applied Probability*, 2011.

[4] N. Jacob: *Pseudo Differential Operators and Markov Processes, Volume I, Markov Processes and Applications*, Imperial College Press, 2001.

[5] N. Jacob: *Pseudo Differential Operators and Markov Processes, Volume II, Markov Processes and Applications*, Imperial College Press, 2002.

[6] N. Jacob: *Pseudo Differential Operators and Markov Processes, Volume III, Markov Processes and Applications*, Imperial College Press, 2005.

[7] X. Kan, H. Qiao and J. Duan: Escape probability for stochastic dynamical systems with jumps. Submitted to *Springer Proceedings in Mathematics (Malliavin Calculus and Stochastic Analysis)*, 2011. arXiv:1111.6419v1 [math.DS]

[8] J. K. Kevorkian and J. D. Cole: *Multiple Scale and Singular Perturbation Methods,* Applied Mathematical Sciences 114, Springer-verlag, New York, 1996.

[9] T. Koren, A. V. Chechkin and J. Klafter: On the first passage time and leapover properties of Lévy motions, *Physica A 379,* (2007)10-22.

[10] M. Liao: The Dirichlet problem of a discontinuous Markov process, *Acta Mathematica Sinica,* 5(1989)9-15.

[11] B. Matkowsky, Z. Schuss, and C. Tier, Asymptotic Methods for Markov Jump Processes, *A. M. S. Lectures in Applied Mathematics* 27, 1991, p. 215-240.

[12] H. Qiao and X. Zhang: Homeomorphism flows for non-Lipschitz stochastic differential equations with jumps, *Stochastic Processes and their Applications,* 118(2008)2254-2268.

[13] K. Sato: *Lévy processes and infinitely divisible distributions*, Cambridge University Press, 1999.

[14] Z. Schuss, *Theory and Applications of Stochastic Differential Equations*, Wiley & Sons, New York, 1980.

[15] H. Tanaka, M. Tsuchiya and S. Watanabe: Perturbation of drift-type for Lévy processes, *J. Math. Kyoto Univ.,* 14(1974)73-92.

[16] W. A. Woyczynski, Lévy processes in the physical sciences. In *Lévy Processes: Theory and Applications*, O. E. Barndorff-Nielsen, T. Mikosch and S. I. Resnick (Eds.), 241-266, Birkhäuser, Boston, 2001.